

MAS317 Linear Algebra II, 2007

Solutions to Coursework 3

1. (i) $\langle v_1, v_2, v_3, v_4 \rangle = \{\text{linear combinations of } v_1, v_2, v_3, v_4\}$. [2 marks]

Let $U = \langle v_1, v_2, v_3, v_4 \rangle$ and $W = \langle v_1, 2v_1 + 5v_2, v_3, v_4 \rangle$.

Then $U \supseteq W$ as all the vectors $v_1, 2v_1 + 5v_2, v_3, v_4$ are in U . [2 marks]

Also $v_2 = (1/5)(2v_1 + 5v_2) + (-2/5)v_1$, and so $W \supseteq U$ as all the vectors v_1, v_2, v_3, v_4 are in W . So $U = W$. [6 marks]

For any a, b by the same argument as above

$\langle v_1, v_2, v_3, v_4 \rangle \supseteq \langle v_1, av_1 + bv_2, v_3, v_4 \rangle$. [2 marks]

To do the converse you need to be able to write v_2 as a linear combination of (some of) $v_1, av_1 + bv_2, v_3, v_4$. If $b = 0$ this may not be possible, but if $b \neq 0$ it is possible, as you can put

$$v_2 = (1/b)(av_1 + bv_2) + (-a/b)v_1.$$

So we require $b \neq 0$ for the two subspaces to be equal. [8 marks]

2. (i) If a linearly independent set contains n vectors in V , then by (2.9)(iii) there is a basis containing these vectors and $n - n$, (that is 0), more vectors. So it is just this set. [5 marks]

(ii) If W is a subspace of V and $\dim W = n$ then W has a basis containing n vectors. This is a linearly independent set of vectors in V and so is a basis for V . So $W = V$. [5 marks]

(iii) If a spanning set for V contains n vectors, then by (2.9)(iii) there is a basis containing n these vectors. So it is all of this set. [5 marks]

3. $U = \langle (1, -3, 4, 2), (1, -4, 11, 0), (-1, 2, 3, -4), (2, -5, 1, 8) \rangle$,

(i) Let $A = \begin{pmatrix} 1 & -3 & 4 & 2 \\ 1 & -4 & 11 & 0 \\ -1 & 2 & 3 & -4 \\ 2 & -5 & 1 & 8 \end{pmatrix}$, so that U is the row space of A .

Doing elementary row operations gives matrices also with row space U . When this gives a matrix B in echelon form then the non-zero rows of B are linearly independent and so are a basis for U .

$$A = \begin{pmatrix} 1 & -3 & 4 & 2 \\ 1 & -4 & 11 & 0 \\ -1 & 2 & 3 & -4 \\ 2 & -5 & 1 & 8 \end{pmatrix} \xrightarrow{\substack{R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 + R_1, R_4 \rightarrow R_4 - 2R_1}} \begin{pmatrix} 1 & -3 & 4 & 2 \\ 0 & -1 & 7 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & 1 & -7 & 4 \end{pmatrix}$$

$$\xrightarrow{\substack{R_3 \rightarrow R_3 - R_2 \\ R_4 \rightarrow R_4 + R_2}} \begin{pmatrix} 1 & -3 & 4 & 2 \\ 0 & -1 & 7 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{pmatrix} 1 & -3 & 4 & 2 \\ 0 & -1 & 7 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = B$$

B is in echelon form, so $(1, -3, 4, 2), (0, -1, 7, -2), (0, 0, 0, 2)$ are a basis for U . **[20 marks]**

(ii) To extend this to a basis for $\mathbb{R}^4$ the row of zeros in B needs to be replaced by a row chosen such that the new matrix is (row equivalent to) a matrix in echelon form with 4 non-zero rows. eg $(0, 0, 1, 0)$ will do, but $(0, 0, 0, 1)$ will not do.

So $(1, -3, 4, 2), (0, -1, 7, -2), (0, 0, 0, 2), (0, 0, 1, 0)$ is a basis for $\mathbb{R}^4$ of the kind required. **[10 marks]**

(iii) If a basis for U consists of some of the vectors u_1, u_2, u_3, u_4 , it will contain 3 vectors as U has a basis with 3 vectors from (i). To find such a basis, note that u_1, u_2, u_3 are linearly dependent as $-2u_1 + u_2 = u_3$.

So one of these three can be left out, so U is spanned by any two of these vectors together with u_4 and this must be a basis (as 3 vectors). (Eg one answer is u_1, u_2, u_4 .) **[10 marks]**

4. (i) the row space does not change by (2.15), and so (ii) the row rank does not change as it is the dimension of the row space, **[5+5 marks]**

(iii) the column space can change, example: **[10 marks]**

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - R_2} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} = B, \text{ and } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \notin C(B).$$

(iv) the column rank does not change by (ii) as it is equal to the row rank by (2.19). **[5 marks]**